TAYLOR SERIES REMAINDER FORMULA

We write down the general formula for Taylor’s series:

Or

In many applications, we do not need the entire Taylor series. We only require a finite number of terms. In this case we write

The question becomes, what is the error in doing this? We would like to know how close this approximation is to the true value. The error is expressed as an integral and it is called the “remainder”.

It is given by

One property of the remainder is that it goes to zero as n goes to infinity.

We want to calculate the remainder for several Taylor series.

In evaluating the remainder, we use a specialized form of the mean value theorem.

This modified theorem states that if g(x) is positive them where c is between a and b.

In this case it becomes where c is in between a and x.

Sometimes it will happen the calculating the derivative fn+1(c) is extremely difficult, so much so that it makes the question intractable. In this case, if the series is alternating, we will be able to use the remainder theorem for alternating series. To reiterate, this theorem says that the remainder is less than the value of the first missing term. So we use the equation R = un+1. This method is relatively easy to use. We will use this and the integral form of the remainder for the questions herein.

**EXAMPLE**

Find the remainder for the Taylor series of ex

Start with the general formula for Taylor series:

If we stop at the nth term this becomes:

The remainder integral is given by

This is a tedious integral. We can evaluate it by using a modified form of the mean value theorem of integral calculus. As such we write

where a < c < x

Integrating we get:

We do not know the exact value for c, although we know it is between x and a.

We can use the fact that to write an inequality:

This is the expression we want. We have an upper bound for the error. All we have to do is plug in x and a. x is the value we plug into the series. a is the point of expansion.

**EXAMPLE**

Find the remainder for the Taylor series of sin x

Start with the general formula for Taylor series:

We can write this series in the somewhat abbreviated way

We write the remainder integral

Using the mean value theorem of integral calculus this becomes:

where a < c < x

For this integral the nth term is present in the series and the (n+1)st term is missing.

If n is even this becomes

ignoring the + or - sign

where a < c < x

If n is odd this becomes

The integration is almost identical and the result will be

ignoring the +/- sign

**EXAMPLE**

Find the remainder for the Taylor series of cos x

Start with the general formula for Taylor series:

We can write this series with the shorthand notation:

We write the remainder integral:

Using the mean value theorem of integral calculus this becomes

where a < c < x

This integral represents the situation where the nth term is present in the series and the n+1 st term is missing.

If n is even this becomes

Ignore the +/- and we write

Using the mean value theorem of integral calculus this becomes

where a < c < x

If n were odd the result would be

**EXAMPLE**

Find the remainder for the Taylor series of expanded about x=0.

First write out Taylors formula:

Or

Starting with the series for

The remainder integral is

Using the mean value theorem of integral calculus this can be written as

where a < c < x

The (n+1)st derivative can be written as . This can be found by actually taking derivatives of ln(1+x) and seeing the pattern.

For us, a = 0 so this becomes:

The remainder is a maximum when c = 0 so this becomes . This is an upper bound for the error. The error must be less that this expression.

**EXAMPLE**

You are given the Taylor series

Find the remainder integral for the Taylor series of ln x expanded about x = a.

Let’s write the series as

From this series we can see that

Write the remainder integral

Modify the integral using the mean value theorem for integrals:

where a < c < x

The remainder is a maximum when c = a so this becomes

This is an upper bound on the error.

**EXAMPLE**

You are given the Maclaurin series

Let x = -1/10.

Find the result accurate to 5 decimal places.

We need the remainder integral to find out how many terms we need in the Maclaurin series.

We write

For us, a = 0 and x = -1/10

Using the mean value theorem for integrals we get

Ignore the -1 in front and set c = 0 to get an upper bound on the error.

We want 5 decimal accuracy so we want set

Rearranging we get

This equation has to be solved numerically:

no good

no good

GOOD!

Calculator answer: 0.904837418

The difference is 0.000004085

**EXAMPLE**

You are given the binomial expression expanded about a = 0. Let x = 0.1.

Use the binomial expansion to find accurate to 3 decimals

We start with the binomial theorem

We write the remainder integral

We use the mean value theorem for integrals to get:

For us, x = 0.1 and a = 0

We need an expression for the (n+1)st derivative.

where we dropped the +/- signs.

Set c = 0. This will give us an upper bound on the error.

We want our result to be accurate to three decimals. The error must be in the 4th decimal place, so we want the error to be less than 0.001

We must solve this numerically:

NO GOOD!

GOOD!

Calculator answer:

The difference is 0.0001085

**EXAMPLE**

Find the square root of 5 accurate to 0.00001

We need the formula for the (n+1)st derivative of the square root function. This was solved in a previous handout. We have:

We now compute the remainder integral:

Using the mean value theorem of integral calculus this becomes:

where a < c < x

We have x = 5 and a = 4 with c in between. To maximize the expression (to get an upper bound for the remainder), we let c = 4. This will give us an upper bound on the error.

This magnitude of this expression must be less than 0.00001

We must solve this numerically:

NO GOOD

NO GOOD

NO GOOD

NO GOOD

NO GOOD (BUT REALLY CLOSE)

GOOD!

We need the Taylor series for the square root centered about x = 4. We will also need the a formula for the (n+1)st derivative.

This was solved previously.

We have

We take the first 6 terms

Calculator answer: 2.236067977

The difference is 0.000001636. We have an extra decimal accuracy. This came about by approximating c with 4.

**EXAMPLE**

Evaluate the integral accurate to 4 decimal places.

This means that the error is in the 5th decimal. So we want the error to be less than 0.00001

We start with the Taylor series for the integrand:

Now that we have the Taylor series, we can evaluate the integral:

Our choice now, to determine the number of terms, is twofold. We can compute a formula for Dn+1(x3 tan-1 x). This derivative is ridiculously difficult. The other choice is to use the remainder theorem for alternating series. This theorem says that the remainder is less than the value of the first neglected term of the series. This is the way we will go.

We want this to be less than 0.00001

Take reciprocals:

This must be solved numerically

NO GOOD

GOOD!

Calculator answer: 0.005915925 Difference is 0.000005454

**EXAMPLE**

Evaluate the integral accurate to 4 decimals

This means that the first 4 decimals must be correct, so the error must be in the 5th decimal so the error

must be < 0.00001. We start with the Taylor series for sin u

Or

Substitute u = x4

We now evaluate the given integral:

We now have to figure out how many terms we need from the series on the right.

Instead of evaluating the remainder integral, we will use the remainder theorem for alternating series. The error in truncating the series is less than the first missing terms – so we set the remainder equal to the first missing term:

ignoring the sign of the term

From the series we see so

We now have

This inequality must be solved numerically:

NO GOOD

G00D!

Calculator answer: 0.1875695447 Difference = 0.000006768

**EXAMPLE**

Evaluate the integral so that it is accurate to 5 X 10-6

We need the binomial theorem

or

For us, n = ½ and u = x4

This yields

Using the double factorial formula (2n)! = (2n-1)!! (2n)!! we transform the above:

Now we use the formula (2n)!! = 2n n!

Rewriting and combining terms this becomes:

This can be written in sigma form:

We can now evaluate the integral

We will use the remainder theorem for alternating series to determine how many terms to take in the series.

The theorem states that the error is less than the first missing term. We set the remainder R = un+1.

This inequality must be solved numerically

GOOD! ON THE FIRST TRY!

Calculator answer: 0.401020391

Difference: 3.609 X 10-6

**EXAMPLE**

Evaluate the integral accurate to within 0.00001

We start with the Taylor series for eu

Substitute u = - x2.

In sigma form this becomes

The integrand can be written as

We can now evaluate the integral:

The derivative is extremely difficult even with Leibnitz rule. Instead we use the fact that the series is alternating and state the remainder is R = un+1

We solve this numerically:

NO GOOD

NO GOOD

G00D!

Calculator answer: 0.0359403074 Difference: 0.000001774

**EXAMPLE**

You are given f(x) = e2x. You expand about a = 0 and you let n=5. Find the sum. Find the remainder. Let x=0.2 and find how close you are to the true answer.

Start with the Taylor series for eu.

Substitute u = 2x

The remainder is given by the integral

a < c < x

There is a formula for the derivative of f(x) = e2x:

We can get an upper bound on the error by setting c = x

At x = 0.2 the remainder is

The series yields:

Calculator answer:

Difference:

**EXAMPLE**

You are given f(x) = cos x. You expand about a = 0 and you let n=8. Find the sum. Find the remainder. Find the value of the approximation when x = 1.

Start with the Taylor series for cos x.

The remainder is given by the integral

a < c < x

The 9th derivative of cosine is negative sine:

The magnitude of the error is given by

We would have gotten a different result had we used the remainder theorem for alternating series. That result would give . Either result should work.

When x = 1 the original form for the remainder is

The series is approximately

Calculator answer: cos 1 = 0.5403023059

Difference: 2.735 X 10-7

**EXAMPLE**

Find the Taylor polynomial for for n = 4 and a = 0. Find the remainder. Find the value of the polynomial at x = 0.5. How close is this to the true answer?

The nth derivative is given by

The remainder integral is

Set c=0 to get an upper bound on the error. Ignore the minus sign.

At x = 0.5 the series is

The remainder is

The true answer is 2/3 = 0.66666…

The difference is 0.020833

**EXAMPLE**

Find the Taylor polynomial up to n = 6 for a = 0; . Find the remainder. Find the value of the Taylor polynomial at x = 1. How close are you to the true answer?

The Taylor polynomial is

There is a formula for

Let c = 1 to get an upper bound on the error.

At n=6 we get

At x = 1 this is

The polynomial equals

The true answer is 2.718281828459045

The difference is 2.263 x 10-4